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CALCULATION OF THE TEMPERATURE DISTRIBUTION IN AN ECCENTRIC ANNULAR LAYER WITH HEAT RELEASE

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A solution is found for the problem of the steady temperature field in a uniform eccentric annular layer with constant internal heat release.

Consider the region included between eccentric circles of radius R_1 and R_2. Let $R_2 > R_1$ and let the centers of the circles x_c lie on the positive branch of the abscissa axis, so that $x_{c_2} > x_{c_1}$. In this notation the eccentricity e of the annulus may be written as

$$e = (x_{c_2} - x_{c_1})/(R_2 - R_1).$$

For this geometry it is necessary to solve the Poisson equation

$$\Delta t(x, y) = -q_v/\lambda \tag{1}$$

when q_v/λ = const.

Let us introduce the function $T(x, y) = t(x, y) + q_v x^2/2\lambda$, for which the equation

$$\Delta T(x, y) = 0 \tag{2}$$

is valid. We shall solve (2) in bipolar coordinates ξ, η, introduced in the form

$$\xi = \{[(x-\alpha)^2 + y^2][(x+\alpha)^2 + y^2]^{-1}\}^{1/2}, \quad 0 \leqslant \xi < 1, \tag{3}$$

$$\eta = \operatorname{arctg} \frac{2\alpha y}{x^2 + y^2 - \alpha^2}, \qquad 0 \leqslant \eta < 2\pi \tag{4}$$

or

$$x = (1 - \xi^2)\alpha/(1 - 2\xi\cos\eta + \xi^2), \tag{5}$$

$$y = 2\alpha\xi\sin\eta/(1 - 2\xi\cos\eta + \xi^2). \tag{6}$$

A description of bipolar coordinates may be found, for example, in [1].

The coordinate lines ξ = const are a family of eccentric circles, whose centers lie on the x axis:

$$x_c = \alpha(1 + \xi^2)/(1 - \xi^2), \quad \xi = [1 + (\alpha/r)^2]^{1/2} - \alpha/r. \tag{7}$$

The coordinate lines η = const are also a family of eccentric circles; their centers lie on the y axis

$$y_c = \alpha \operatorname{ctg}\eta, \quad r = \alpha/|\sin\eta|. \tag{8}$$

Knowing R_1, R_2, and e, we can choose α so that the boundaries of the annulus coincide with the coordinate lines ξ = const:

$$\alpha = (1/2e)\{[R_1(1+e^2) + R_2(1-e^2)]^2 - 4e^2 R_1^2\}^{1/2} =$$
$$= (1/2e)\{[R_1(1-e^2) + R_2(1+e^2)]^2 - 4e^2 R_2^2\}^{1/2}. \tag{9}$$

It should be noted that when e = 1 the bipolar coordinate system loses its meaning, since then α = 0 and by (3) any point on the plane (x, y) has coordinates ξ = 1, η = 0. In bipolar coordinates Eq. (2) may be written as

$$\frac{1}{\alpha^2}\left(\frac{1+\xi^2}{2\xi} - \cos\eta\right)^2 \left[\xi^2 \frac{\partial^2 T}{\partial \xi^2} + \xi \frac{\partial T}{\partial \xi} + \frac{\partial^2 T}{\partial \eta^2}\right] = 0 \tag{10}$$

or

$$\xi^2 \frac{\partial^2 T}{\partial \xi^2} + \xi \frac{\partial T}{\partial \xi} + \frac{\partial^2 T}{\partial \eta^2} = 0.$$

The general solution of this equation may be found by the method of separation of variables [2]:

$$T(\xi, \eta) = A_0 + B_0 \ln \xi + \sum_{n=1}^{\infty}[(A_n \xi^n + B_n \xi^{-n}) \cos n\eta + (C_n \xi^n + D_n \xi^{-n}) \sin n\eta], \tag{11}$$

$$t(\xi, \eta) = T(\xi, \eta) - \frac{q_v}{2\lambda} x^2(\xi, \eta). \tag{12}$$

Using the Fourier expansion of $x^2(\xi, \eta)$ (an even function with respect to η) [3], we write

$$t(\xi, \eta) = A_0 + B_0 \ln \xi - \frac{a_0(\xi)}{2} + \sum_{n=1}^{\infty}\{[A_n \xi^n + B_n \xi^n - a_n(\xi)] \cos n\eta +$$

$$+ [C_n \xi^n + D_n \xi^{-n}] \sin n\eta\}, \tag{13}$$

where

$$a_n(\xi) = \frac{1}{\pi}\int_0^{2\pi} \frac{q_v \alpha^2}{2\lambda}\left[\frac{1-\xi^2}{1-2\xi\cos\eta+\xi^2}\right]^2 \cos n\eta \, d\eta =$$

$$= \frac{q_v \alpha^2(1-\xi^2)^2}{\pi\lambda}\int_0^{\pi}\frac{\cos n\eta \, d\eta}{(1-2\xi\cos\eta+\xi^2)^2} = \frac{q_v a^2 \xi^2}{\varkappa+\xi^2}[(n+1)-(n-1)\xi^2] \tag{14}$$

$$(n = 0, 1 \ldots).$$

In the case of boundary conditions of the first kind, i.e., when the temperature on the boundary of the region is given, the unknown coefficients A_0, B_0, A_n, B_n, are determined as in the case of a concentric annulus. If the given boundary conditions are of the second or third kind, then we must bear in mind that

$$\frac{\partial t}{\partial n} = \frac{1}{2a}(1+\xi^2-2\xi\cos\eta)\frac{\partial t}{\partial \xi},$$

i.e., the determination of the unknown coefficients is more complicated.

To illustrate the method, we shall find the temperature distribution in a cylinder with heat release covered with an eccentric, annular, thermally insulating layer, the outer boundary of which is maintained at constant temperature t_0.

In accordance with (16) we write the solution:
a) in the cylinder

$$t_1(\xi, \eta) = A_0 - \frac{a_0(\xi)}{2} + \sum_{n=1}^{\infty}[A_n \xi^n - a_n(\xi)]\cos n\eta; \tag{15}$$

b) in the covering

$$t_2(\xi, \eta) = B_0 + C_0 \ln \xi + \sum_{n=1}^{\infty}[B_n \xi^n + C_n \xi^{-n}] \cos n\eta. \tag{16}$$

Using the boundary condition $t_2(\xi_2, \eta) = t_0$, we obtain

$$t_2(\xi, \eta) = B_0\left(1 - \frac{\ln \xi}{\ln \xi_2}\right) + t_0 \frac{\ln \xi}{\ln \xi_2} + \sum_{n=1}^{\infty} B_n [\xi^n - \xi^{-n} \xi_2^{2n}] \cos n\eta. \tag{17}$$

From the joining conditions

$$t_1(\xi_1, \eta) = t_2(\xi_1, \eta), \quad \lambda_1 \left.\frac{\partial t_1}{\partial n}\right|_{\xi=\xi_1} = \lambda_2 \left.\frac{\partial t_2}{\partial n}\right|_{\xi=\xi_1} \tag{18}$$

we find all the unknown coefficients:

$$A_0 = t_0 + \frac{a_0(\xi_1)}{2} + \frac{\lambda_1}{2\lambda_2}\xi_1 \ln\frac{\xi_2}{\xi_1} a_0'(\xi_1),$$

$$B_0 = t_0 + \frac{\lambda_1}{2\lambda_2}\xi_1 \ln \xi_2 a_0'(\xi_1),$$

$$A_n = \{\lambda_1[a_n'(\xi_1)/n]\xi_1[1+(\xi_2/\xi_1)^{2n}] - \lambda_2 a_n(\xi_1)[1+(\xi_2/\xi_1)^{2n}]\}\times$$
$$\times\{\lambda_1\xi_1^n[1-(\xi_2/\xi_1)^{2n}] - \lambda_2\xi_1^n[1+(\xi_2/\xi_1)^{2n}]\}^{-1},$$
$$C_n = \{\lambda_1[a_n'(\xi_1)/n]\xi_1 - \lambda_1 a_n(\xi_1)\}\times$$
$$\times\{\lambda_1\xi_1^n[1-(\xi_2/\xi_1)^{2n}] - \lambda_2\xi_1^n[1+(\xi_2/\xi_1)^{2n}]\}^{-1}, \tag{19}$$

where

$$a_n'(\xi_i) = \frac{da_n}{d\xi}\bigg|_{\xi=\xi_i}.$$

For the specific case ($R_1 = 0{,}5$; $R_2 = 5$ mm; $e = 0{,}5$; $t_0 = 0^\circ$ C; $q_v/\lambda = 10^7$ m $^{-2}$; $\lambda_1 = \lambda_2$) we obtain

$$A_0 - a_0(\xi_1)/2 = 0.04600\, q_v\alpha^2/\lambda,$$

$$A_1\xi_1 - a_1(\xi_1) = 0.01005\, q_v\alpha^2/\lambda,$$

$$A_2\xi_1^2 - a_2(\xi_1) = 0.00123\, q_v\alpha^2/\lambda,$$

$$A_3\xi_1^3 - a_3(\xi_1) = 0.000083\, q_v\alpha^2/\lambda,$$

i.e., the term with n = 3 contributes ~0.13% to the result. The temperature distribution along the axis Ox for e = 0.5 and e = 0.0 is shown in the figure.

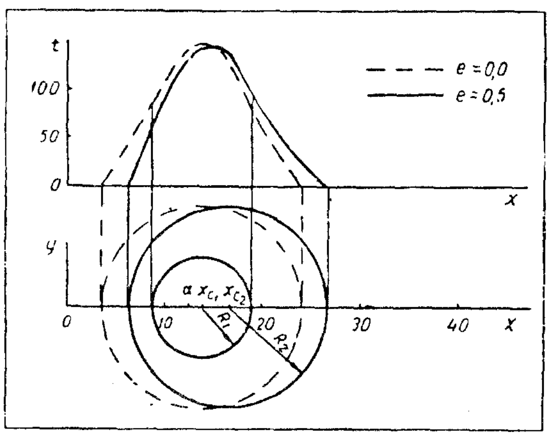

Fig. Comparison of temperature distributions in a heat-generating cylinder covered with an annular layer of insulator for two different values of the eccentricity e.

When using the above method to calculate the temperature at $e \ll 1$ ($\alpha \gg 1$) difficulties may arise which are easily removed by changing from the variable ξ to the variable r:

$$\xi = \sqrt{1+\left(\frac{\alpha}{r}\right)^2} - \frac{\alpha}{r} \cong \frac{1}{2}\frac{r}{\alpha}. \tag{20}$$

Then, for example, the solution of the problem of an annulus whose boundary is kept at temperature t = 0 is written as

$$t(r,\eta) \cong \frac{q_v}{4\lambda}\Big\{(R_2^2 - R_1^2)\times$$
$$\times\frac{\ln(r/R_1)}{\ln(R_2/R_1)} - (r^2 - R_1^2) +$$
$$+\frac{r}{\alpha}\left[R_1^2 + R_2^2 - \left(\frac{R_1R_2}{r}\right)^2 - r^2\right]\cos\eta\Big\}. \tag{21}$$

From (6) and (7) we obtain

$$\sin\varphi = (1-\xi^2)\sin\eta/(1-$$
$$-2\xi\cos\eta + \xi^2) \cong \sin\eta, \tag{22}$$

i.e., the usual polar coordinate φ may be written instead of η in (21) when $\xi \ll 1$.

The last inequality is satisfied when e ≪ 1. If $\alpha = \infty$ (e = 0), Eq. (21) goes over into the formula for the temperature field in an annulus formed by concentric circles.

Finally, it should be noted that the method described allows the steady temperature field to be found inside an eccentric annulus with heat release for various kinds of boundary condition. This method must not be applied when the eccentricity is equal or close to unity, and at eccentricities close to zero, we must convert from the variable ξ to the variable r (20).

NOTATION

r, φ—polar coordinates; x, y and ξ, η—rectangular and bipolar coordinates; x_{c_i}—coordinate of center of i-th circle; e—eccentricity of annulus; t, λ, q_V—temperature, thermal conductivity, and internal heat release, respectively.

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